# RESEARCH STATEMENT

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My research interests encompass two broad areas of analysis. The first area is in the intersection of geometric function theory, nonlinear potential theory and analysis on metric spaces. The second area consists of operator theoretic aspects of function theory, especially corona and interpolation problems. In particular, in the first area, I have been studying sufficient conditions under which one would get strong $A_\infty$-weights in $\mathbf{R}^n$ and in metric measure spaces. I am also interested in Sobolev-Lorentz capacity and Besov capacity. I explore how Hausdorff measures and each of these capacities are related. More recently, in the first area I have been studying Newtonian Lorentz metric spaces. In the second area I have been studying corona and interpolation problems for Hilbert function spaces including the Drury-Arveson Hardy space, a space that occupies a central position in multivariate operator theory.

## 1. Current Research

We consider separately the two main areas, beginning with the area of operator function theory.

1.1. **Operator function theory.** In 1962 L. Carleson proved in [Car] his famous corona theorem for $H^\infty(\mathbf{D})$, the Banach algebra of bounded analytic functions in the unit disk $\mathbf{D}$. He showed that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbf{D})$ satisfying

$$\sum_{j=1}^N |g_j(z)| \geq c > 0, \qquad z \in \mathbf{D}, \tag{1}$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbf{D})$ with

$$\sum_{j=1}^N f_j(z)\, g_j(z) = 1, \qquad z \in \mathbf{D}. \tag{2}$$

This means there is no corona in the maximal ideal space $\mathcal{M}$ of $H^\infty(\mathbf{D})$, i.e. point evaluations are dense in $\mathcal{M}$. We observe that $H^\infty(\mathbf{D})$ is the multiplier algebra of the classical Hardy space $H^2(\mathbf{D})$.

Later, Hörmander noted a connection between the corona problem and the Koszul complex, and in the late 1970's Tom Wolff gave a simplified proof using the theory of the $\overline{\partial}$ equation and Green's theorem (see [Gar]). While there is a large literature on such corona theorems in one complex dimension (see e.g. [Nik]), progress in higher dimensions has been limited until recently. Indeed, apart from the simple cases in which the maximal ideal space of the algebra can be identified with a compact subset of $\mathbf{C}^n$, no corona theorem for a multiplier algebra has been proved in higher dimensions until recently. Instead, partial results have been obtained, such as the $H^p$ corona theorem on the ball and polydisk, and results restricting $N$ to 2 generators in (1) (the case $N = 1$ is trivial). (See Andersson-Carlsson [AC1, AC2, AC3], Treil-Wick [TW], Krantz-Li [KL], Lin [Lin], Trent [Tre], Amar [Ama] and Ortega-Fabrega [OF]).

In recent work with E. Sawyer and B. Wick (see [CSW2]) we have obtained the corona theorem for the multiplier algebra of the Drury-Arveson Hardy space $H_n^2$ with infinitely many generators. This is the first generalization of Carleson's Corona theorem to a multiplier algebra of holomorphic functions in *higher dimensions*. The key to this result is the following theorem on the "baby corona problem".

**Theorem 1.** *Let $\delta > 0$, $\sigma \geq 0$ and $1 < p < \infty$. Then there is a constant $C_{n,\sigma,p,\delta}$ such that given $g = (g_i)_{i=1}^{\infty}$ in $M_{B_p^\sigma(\mathbf{B}_n)\to B_p^\sigma(\mathbf{B}_n;\ell^2)}$ satisfying*

$$\begin{aligned} \|M_g\|_{B_p^\sigma(\mathbf{B}_n)\to B_p^\sigma(\mathbf{B}_n;\ell^2)} &\leq 1, \\ \sum_{j=1}^{\infty} |g_j(z)|^2 &\geq \delta^2 > 0, \qquad z \in \mathbf{B}_n, \end{aligned}$$

*there is for each $h \in B_p^\sigma(\mathbf{B}_n)$ a vector-valued function $f \in B_p^\sigma\left(\mathbf{B}_n;\ell^2\right)$ satisfying*

$$\begin{aligned} \|f\|_{B_p^\sigma(\mathbf{B}_n;\ell^2)} &\leq C_{n,\sigma,p,\delta}\|h\|_{B_p^\sigma(\mathbf{B}_n)}, \\ \sum_{j=1}^{\infty} g_j(z) f_j(z) &= h(z), \qquad z \in \mathbf{B}_n. \end{aligned}$$

We recall that $B_p^\sigma(\mathbf{B}_n)$ consists of all $f \in H(\mathbf{B}_n)$ such that

$$\|f\|_{B_p^\sigma(\mathbf{B}_n)} \equiv \sum_{k=0}^{m-1}\left|\nabla^k f(0)\right| + \left(\int_{\mathbf{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma}\nabla^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} < \infty, \tag{3}$$

for some $m > \frac{n}{p} - \sigma$. Here $\lambda_n$ is the invariant measure on the ball $\mathbf{B}_n$. The vector-valued version $B_p^\sigma\left(\mathbf{B}_n;\ell^2\right)$ is defined in the usual way. It is known that the right side is finite for some $m > \frac{n}{p} - \sigma$ if and only if it is finite for all $m > \frac{n}{p} - \sigma$.

If we now invoke the Toeplitz corona theorem, we obtain the corona theorem for the Drury-Arveson space $H_n^2 = B_2^{\frac{1}{2}}(\mathbf{B}_n)$.

**Corollary 2.** *Let $0 \leq \sigma \leq \frac{1}{2}$. Then the Banach algebra $M_{B_2^\sigma(\mathbf{B}_n)}$ has no corona, i.e. the linear span of point evaluations $e_z(f) = f(z)$, $f \in M_{B_2^\sigma(\mathbf{B}_n)}$ and $z \in \mathbf{B}_n$, is dense in the maximal ideal space of $M_{B_2^\sigma(\mathbf{B}_n)}$. In particular the multiplier algebra of the Drury-Arveson space $H_n^2$ has no corona.*

Thus we settle the Corona problem for the Drury-Arveson space, which from the point of view of operator theory is the correct generalization of the Hardy space to several variables. This was a question that had been open for some time in this area. Moreover, this is the first positive Corona theorem in several dimensions that we are aware of.

In [CSW1] we generalize both a theorem of Varopoulos [Var] and a theorem of Andersson-Carlsson [AC2] by obtaining $BMOA(\mathbf{B}_n)$ solutions to the $H^\infty(\mathbf{B}_n)$ Corona problem with infinitely many generators.

**Theorem 3.** *There is a constant $C_{n,\delta}$ such that given $g = (g_i)_{i=1}^{\infty} \in H^\infty(\mathbf{B}_n;\ell^2)$ satisfying*

$$1 \geq \sum_{j=1}^{\infty}|g_j(z)|^2 \geq \delta^2 > 0, \qquad z \in \mathbf{B}_n, \tag{4}$$

*there is for each $h \in H^\infty(\mathbf{B}_n)$ a vector-valued function $f \in BMOA\left(\mathbf{B}_n;\ell^2\right)$ satisfying*

$$\begin{aligned} \|f\|_{BMOA(\mathbf{B}_n;\ell^2)} &\leq C_{n,\delta}\|h\|_{H^\infty(\mathbf{B}_n)}, \\ \sum_{j=1}^{\infty} f_j(z) g_j(z) &= h(z), \qquad z \in \mathbf{B}_n. \end{aligned}$$

We recall that $BMOA(\mathbf{B}_n) = BMO(\partial\mathbf{B}_n) \cap H^2(\mathbf{B}_n)$, where $BMO(\partial\mathbf{B}_n)$ is the collection of functions in $L^2(\partial\mathbf{B}_n)$ such that

$$\|b\|_{BMO(\partial\mathbf{B}_n)}^2 \equiv \sup_{Q_\delta(\eta)\subset\partial\mathbf{B}_n}\frac{1}{|Q_\delta(\eta)|}\int_{Q_\delta(\eta)}|b - b_{Q_\delta(\eta)}|^2 d\sigma(\zeta) < \infty.$$

Here $Q_\delta(\eta)$ is the non-isotropic ball of radius $\delta > 0$ and center $\eta$ in $\partial\mathbf{B}_n$ and

$$b_{Q_\delta(\eta)} = \frac{1}{|Q_\delta(\eta)|}\int_{Q_\delta(\eta)} b(\xi)d\sigma(\xi).$$

The vector-valued versions $H^\infty(\mathbf{B}_n;\ell^2)$, $H^2(\mathbf{B}_n;\ell^2)$, $BMO(\partial\mathbf{B}_n;\ell^2)$, and $BMOA(\mathbf{B}_n;\ell^2)$ are defined in the usual way.

1.2. **Geometric function theory, nonlinear potential theory and analysis on metric spaces.** We begin with some work on the strong $A_\infty$-weights of David and Semmes; then we describe the work done on Besov capacity and Sobolev-Lorentz capacity, some of which is joint with Maz'ya. We finish by describing the work done with Miranda Jr. on Newtonian Lorentz metric spaces.

Suppose $(X, d, \mu)$ is a complete and unbounded geodesic Ahlfors $Q$-regular metric measure space for some $Q > 1$. A space $(X, d, \mu)$ is called Ahlfors $Q$-regular if the $\mu$-measure of balls of radii $r$ is comparable to $r^Q$.

A nontrivial doubling measure $\nu$ on $X$ is a Radon measure for which there exists a constant $C > 1$ such that $0 < \nu(2B) \le C\nu(B)$ for all balls $B$; here $2B$ is the ball concentric with $B$ but twice its radius. Given a doubling measure $\nu$ on $X$, one can define an associated quasidistance on $X$ by

$$\delta_\nu(x, y) = \nu(B_{x,y})^{1/Q}, \tag{5}$$

where $B_{x,y} = B(x, d(x, y)) \cup B(y, d(x, y))$. To say that $\delta_\nu(x, y)$ is a quasidistance means that it is nonnegative and symmetric, that it vanishes exactly when $x = y$, and that it satisfies the following weakened form of the triangle inequality:

$$\delta_\nu(x, z) \le C(\delta_\nu(x, y) + \delta_\nu(y, z))$$

for some $C \ge 1$ and all $x, y, z \in X$. If the above inequality was true with $C = 1$, then $\delta_\nu$ would be a distance function (as opposed to a quasidistance).

A doubling measure $\nu$ on $X$ is said to be a *metric doubling measure* if the quasidistance $\delta_\nu$ is comparable to a distance $\delta'_\nu$, namely there exists a distance function $\delta'_\nu$ on $X$ and a constant $C \ge 1$ such that

$$C^{-1}\delta_\nu(x, y) \le \delta'_\nu(x, y) \le C\delta_\nu(x, y)$$

for all $x, y \in X$.

A weight $w$ is said to be an $A_\infty$-*weight* with respect to the measure $\mu$ and we write $w \in A_\infty(\mu)$ if there exist constants $C \ge 1$ and $p \in (1, \infty)$ such that the following reverse Hölder inequality holds on all balls $B \subset X$ :

$$\left(\frac{1}{\mu(B)}\int_B w(x)^{-1/(p-1)}\,d\mu(x)\right)^{p-1}\frac{1}{\mu(B)}\int_B w(x)\,d\mu(x) \le C.$$

To say that $w$ is a *strong* $A_\infty$-*weight* means, by definition, that $w$ is an $A_\infty$-weight and that there exists a metric doubling measure $\nu$ on $X$ with density $w$; that is, $\nu(A) = \int_A w(x)\,d\mu(x)$ for all measurable sets $A \subset X$ and $\nu$ is a metric doubling measure.

Strong $A_\infty$-weights in $\mathbf{R}^n$ were introduced in the early 90's by David and Semmes in [DS] and [Sem] when trying to identify the subclass of $A_\infty$-weights that are comparable to the Jacobian determinants of quasiconformal mappings.

It is known that the quasiconformal Jacobians are strong $A_\infty$-weights. It has been proved by Bishop in [Bis] that there exist strong $A_\infty$-weights which are not comparable to any quasiconformal Jacobian.

In the last few years strong $A_\infty$-weights were studied by Bonk-Lang in [BL], Bonk-Heinonen-Saksman in [BHS1] and [BHS2] in the Euclidean setting, and by Korte-Maasalo [KoM] in the metric setting. I also studied strong $A_\infty$-weights in the Euclidean setting in [Cos1] and in the metric setting in [Cos5].

Bonk and Lang proved in [BL] that if $\nu_0$ is a signed Radon measure on $\mathbf{R}^2$ such that $\nu_0^+(\mathbf{R}^2) < 2\pi$ and $\nu_0^-(\mathbf{R}^2) < \infty$, then $w = e^{2u}$ is comparable to the Jacobian of a quasiconformal mapping

$f : \mathbf{R}^2 \to \mathbf{R}^2$, which implies that $w$ is a strong $A_\infty$-weight. Here $u$ is a solution of $-\Delta u = \nu_0$ with $|\nabla u| \in L^2(\mathbf{R}^2)$, while $\nu_0 = \nu_0^+ - \nu_0^-$ is the Jordan decomposition of the signed measure $\nu_0$ into its positive and negative parts.

I have proved a related result in the Euclidean setting (see [Cos1]) and its analogue in Ahlfors $Q$-regular metric spaces that satisfy a weak $(1, s)$-Poincaré inequality (see [Cos5]).

**Theorem 4.** (See [Cos1, Theorem 5.1].) *Let $s \in (n-1, n)$. There exists $\varepsilon = \varepsilon(n, s) > 0$ such that if $u \in L^1_{loc}(\mathbf{R}^n)$ with distributional gradient $\nabla u$ in the Morrey-Campanato space $\mathcal{L}^{s,n-s}(\mathbf{R}^n)$ with the associated norm less than $\varepsilon$, then $w = e^{nu}$ is a strong $A_\infty$-weight with data depending only on $n$ and $s$.*

Recall that for $1 \le p \le n$, the *Morrey space* $\mathcal{L}^{p,n-p}(\mathbf{R}^n) = \mathcal{L}^{p,n-p}(\mathbf{R}^n, m_n)$ is defined to be the linear space of Lebesgue measurable functions $u \in L^1_{loc}(\mathbf{R}^n)$ such that $||u||_{\mathcal{L}^{p,n-p}(\mathbf{R}^n)} < \infty$, where

$$||u||_{\mathcal{L}^{p,n-p}(\mathbf{R}^n)} = \sup_{r>0} \sup_{x \in \mathbf{R}^n} \left( r^p \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)|^p dm_n(y) \right)^{1/p}.$$

In particular, $\mathcal{L}^{n,0}(\mathbf{R}^n) = L^n(\mathbf{R}^n)$. Here $m_n$ is the $n$-dimensional Lebesgue measure in $\mathbf{R}^n$. See for example [Gia, p. 65] for more information about Morrey spaces and their use in the theory of partial differential equations. Similarly one can define the Morrey space $\mathcal{L}^{p,\lambda}(\mathbf{R}^n; \mathbf{R}^m)$ for vector-valued measurable functions.

It is proved in [BHS1, Theorem 3.1] that if $u$ belongs to the Bessel potential space $L^{\alpha,\frac{n}{\alpha}}(\mathbf{R}^n)$, $0 < \alpha < n$, then $w = e^{nu}$ is a strong $A_\infty$-weight with data depending only on $\alpha, n$, and the $L^{\alpha,\frac{n}{\alpha}}$-norm of $u$. The function space $L^{\alpha,p}(\mathbf{R}^n)$ consists of all functions $u$ with "$\alpha$ derivatives" in $L^p(\mathbf{R}^n)$. Recall that for $\alpha > 0$ and $1 < p < \infty$, the Bessel potential space is $L^{\alpha,p}(\mathbf{R}^n) = G_\alpha * L^p(\mathbf{R}^n)$, where $G_\alpha$ is the Bessel kernel of order $\alpha$ defined via its Fourier transform

$$\hat{G}_\alpha(\eta) = (1 + |\eta|^2)^{-\frac{\alpha}{2}}.$$

By the well-known theorem of Calderón, $L^{\alpha,p}(\mathbf{R}^n)$ coincides with the Sobolev space $W^{\alpha,p}(\mathbf{R}^n)$ if $\alpha$ is a positive integer. See for example [AH, Chapter 1] and references therein for discussion.

I have demonstrated in [Cos0] and [Cos1] a result analogous to [BHS1, Theorem 3.1]:

**Theorem 5.** (See [Cos1, Theorem 5.2].) *Let $p \in (n, \infty)$. There exists $\varepsilon = \varepsilon(n, p) > 0$ such that if $u \in L^1_{loc}(\mathbf{R}^n)$ is in the Besov space $B_p(\mathbf{R}^n)$ with the associated Besov seminorm less than $\varepsilon$, then $w = e^{nu}$ is a strong $A_\infty$-weight with data depending only on $n$ and $p$.*

Recall that for $1 < n < p < \infty$ the Besov seminorm $[\cdot]_{B_p(\mathbf{R}^n)}$ is given by

$$[u]_{B_p(\mathbf{R}^n)} = \left( \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{2n}} \, dx \, dy \right)^{1/p}.$$

It is known that $L^{\frac{n}{p},p}(\mathbf{R}^n) \subset B_p(\mathbf{R}^n)$ for every $p \in (n, \infty)$, so [Cos1, Theorem 5.2] generalizes [BHS1, Theorem 3.1] to Besov $B_p$ spaces, where $n < p < \infty$. Besov spaces have recently been used in the study of the quasiconformal mappings in metric spaces and in geometric group theory by Bourdon and Pajot. See [Bou] and [BP].

While studying these function spaces, I was led to develop a theory of Besov $B_p$-capacity in $\mathbf{R}^n$ and in Ahlfors $Q$-regular metric spaces with $Q > 1$. Capacities associated with Besov spaces were studied by Netrusov in [Net1] and [Net2] and by Adams and Hurri-Syrjänen in [AHS].

For $p \in (n, \infty)$, $\Omega \subset \mathbf{R}^n$ bounded and open and $K \subset \Omega$ compact, one defines the *relative Besov p-capacity* of the pair $(K, \Omega)$ by

$$\mathrm{cap}_{B_p}(K, \Omega) = \inf\{[u]^p_{B_p(\mathbf{R}^n)} : u \in C_0^\infty(\Omega) \text{ with } u \ge 1 \text{ on a neighborhood of } K\}$$

and extends the definition to arbitrary sets $E \subset \Omega$. I prove that this capacity is a Choquet set function and relate Hausdorff measures to Besov capacity. Lower estimates of the Besov relative capacity are obtained in terms of the Hausdorff content $\Lambda_h^\infty$ associated with gauge functions $h$. Specifically (see [Cos0] and [Cos1, Theorem 3.8]):

**Theorem 6.** *Let $p \in (n, \infty)$ be fixed. Suppose $h : [0, \infty) \to [0, \infty)$ is an increasing homeomorphism such that $t \mapsto h(t)t^{-n}$, $0 < t < \infty$ is decreasing. There exists a positive constant $C_1 = C_1(n, p)$ such that*

$$\frac{\Lambda_h^\infty(E \cap B(x, 2^{-k}r))}{\left(\int_0^{2^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \leq C_1 k^{p-1} \mathrm{cap}_{B_p}(E \cap B(x, 2^{-k}r), B(x, r))$$

*for every $E \subset \mathbf{R}^n$, every integer $k > 1$, every $x \in \mathbf{R}^n$, and every $r > 0$, where $p'$ is the Hölder conjugate of $p$.*

I also obtained sharp Besov capacity estimates. (See [Cos0] and [Cos1, Theorem 3.11]). Namely, I proved the following theorem:

**Theorem 7.** *Let $p \in (n, \infty)$ be fixed. There exists $C_0 = C_0(n, p) > 0$ such that*

$$\frac{1}{C_0}\left(\ln\frac{R}{r}\right)^{1-p} \leq \mathrm{cap}_{B_p}(B(x, r), B(x, R)) \leq C_0\left(\ln\frac{R}{r}\right)^{1-p}$$

*for every $x \in \mathbf{R}^n$ and every pair of numbers $r, R$ such that $0 < r < \frac{R}{2}$.*

I have also found a sufficient condition to get sets of Besov relative capacity zero, namely (see [Cos1, Theorem 3.16]):

**Theorem 8.** *Let $p \in (n, \infty)$ be fixed and $h : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism such that $h(t) = (\ln \frac{1}{t})^{1-p}$ for all $t \in (0, \frac{1}{2})$. Then for compact sets $E \subset \mathbf{R}^n$, we have that $\Lambda_h(E) < \infty$ implies $\mathrm{cap}_{B_p}(E, \Omega) = 0$ whenever $\Omega \subset \mathbf{R}^n$ is a bounded and open neighborhood of $E$.*

The notions about Besov $p$-capacity discussed before make sense also in a metric space setting. In [Cos3], under the assumption that $(X, d, \mu)$ is a complete and unbounded $Q$-Ahlfors metric measure space for some $Q > 1$, I develop a theory of Besov capacity for $Q < p < \infty$. For every $p$ in $(Q, \infty)$ one can define the Besov space $B_p(X)$ to be the collection of all functions $u \in L^p(X, \mu)$ with finite Besov $p$-seminorm given by

$$[u]_{B_p(X)} = \left(\int_{X\times X} \frac{|u(x) - u(y)|^p}{d(x, y)^{2Q}} d\mu(x)\, d\mu(y)\right)^{1/p}.$$

Recall that a metric measure space $(X, d, \mu)$ of Hausdorff dimension $Q$ is called $Q$-Ahlfors-regular if the $\mu$-measure of balls of radii $r$ is comparable to $r^Q$.

For $\Omega \subset X$ bounded and $K \subset \Omega$ compact, one defines the *relative Besov p-capacity* of the pair $(K, \Omega)$ by

$$\mathrm{cap}_{B_p}(K, \Omega) = \inf\{[u]^p_{B_p(X)} : u \in Lip_0(\Omega) \text{ with } u \geq 1 \text{ on a neighborhood of } K\},$$

where $Lip_0(\Omega)$ is the set of real-valued Lipschitz functions defined on $X$ compactly supported in $\Omega$. This definition can be extended as in [Cos0] and [Cos1] to arbitrary sets $E \subset \Omega$. I prove that this capacity is Choquet. I obtain results similar to those from [Cos0] and [Cos1]. I obtain a sufficient condition to get sets of Besov relative capacity zero as in Theorem 8.

Another capacity that was of interest to me was the Sobolev-Lorentz $p, q$-capacity, which is defined using the Lorentz norm. Thus we generalized the Sobolev $p$-capacity, which is the Sobolev-Lorentz one when $p = q$.

Let $(\Omega, \nu)$ be a measure space where $\Omega \subset \mathbf{R}^n$ is an open set. Let $u : \Omega \mapsto \mathbf{R}^m, m \geq 1$ be a $\nu$-measurable function. Let $u_\nu^*$ be the nonincreasing rearrangement of $u$ with respect to the measure $\nu$. The Lorentz space $L^{p,q}(\Omega, \nu; \mathbf{R}^m), 1 < p < \infty, 1 \leq q \leq \infty$ is the collection of all $\nu$-measurable functions $f : \Omega \to \mathbf{R}^m$ such that

$$||f||_{L^{p,q}(\Omega,\nu;\mathbf{R}^m)} = ||\,|f|\,||_{L^{p,q}(\Omega,\nu)} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f_\nu^*(t))^q \, \frac{dt}{t}\right)^{\frac{1}{q}} & 1 \le q < \infty \\ \sup_{s>0} s^{\frac{1}{p}} f_\nu^*(s) & q = \infty \end{cases}$$

is finite. We notice that $L^{n,q}(\mathbf{R}^n, m_n) \hookrightarrow L^{n,\infty}(\mathbf{R}^n, m_n) \hookrightarrow \mathcal{L}^{p,n-p}(\mathbf{R}^n, m_n)$ whenever $1 \le p < n < q \le \infty$. Lorentz spaces have been studied extensively by Bennett and Sharpley in [BS]. Sobolev-Lorentz spaces have recently been studied by Kauhanen, Koskela, and Malý in [KKM] and by Malý, Swanson, and Ziemer in [MSZ]. Capacities related to Morrey spaces were studied by Adams and Xiao in [AX]. I studied the Sobolev-Lorentz $n,q$-capacity in [Cos2] and the $p,q$-capacity in joint work with V. Maz'ya in [CoMa] when $\nu$ was the $n$-dimensional Lebesgue measure $m_n$.

Let $\Omega \subset \mathbf{R}^n$ be open let $K \subset \Omega$ be compact. Suppose $1 < p < \infty$ and $1 \le q \le \infty$. The Sobolev-Lorentz $p,q$-capacity of the pair $(K, \Omega)$ is defined by

$$\mathrm{cap}_{p,q}(K,\Omega) = \inf\{||\nabla u||^p_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} : u \in C_0^\infty(\Omega) \text{ with } u \ge 1 \text{ on a neighborhood of } K\}.$$

My results from [Cos2] concerning the Sobolev-Lorentz $n,q$-capacity generalize some of the results concerning $s$-capacity on $\mathbf{R}^n$ for $s \in (1,n]$. See [HKM, Chapter 2] for the $s$-capacity on $\mathbf{R}^n$.

For the case $1 \le q \le \infty$, I have found sharp estimates for the $n,q$-capacity of pairs $(\overline{B}(0,r), B(0,1))$ for small radii $r$ of a form similar to that in Theorem 7. Specifically (see [Cos2, Theorem 3.11]):

**Theorem 9.** *Suppose that $1 \le q \le \infty$ and let $q'$ be the Hölder conjugate of $q$. There exists a constant $C(n,q) > 0$ such that*

$$C(n,q)^{-1}\left(\ln\frac{1}{r}\right)^{-n/q'} \le \mathrm{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \le C(n,q)\left(\ln\frac{1}{r}\right)^{-n/q'}$$

*for every $0 < r < \frac{1}{2}$.*

In the case $q = 1$, Theorem 9 implies the existence of a constant $C(n) > 0$ such that $\mathrm{cap}_{n,1}(\{x\},\Omega) = C(n)$ whenever $x \in \mathbf{R}^n$ and $\Omega$ is an open subset of $\mathbf{R}^n$ containing $x$.

The $n,q$-capacity and Hausdorff measures are related. I proved the following theorem (see [Cos2, Theorem 4.4]):

**Theorem 10.** *Suppose that $1 < q \le \infty$ and let $q'$ be the Hölder conjugate of $q$. Suppose that $K$ is a compact set in $\mathbf{R}^n$. Let $h : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism such that $h(t) = \left(\ln\frac{1}{t}\right)^{-\frac{n}{q'}}$ for all $t \in (0, \frac{1}{2})$.*

(i) *If $q \in (1,n)$, then $\Lambda_{h^{q/n}}(K) < \infty$ implies $\mathrm{cap}_{n,q}(K,\Omega) = 0$ whenever $\Omega$ is an open neighborhood of $E$.*

(ii) *If $q \in [n,\infty)$, then $\Lambda_h(K) < \infty$ implies $\mathrm{cap}_{n,q}(K,\Omega) = 0$ whenever $\Omega$ is an open neighborhood of $E$.*

(iii) *If $q = \infty$, then $\Lambda_h(K) = 0$ implies $\mathrm{cap}_{n,\infty}(K,\Omega) = 0$ whenever $\Omega$ is an open neighborhood of $E$.*

These results are Sobolev-Lorentz analogues of those obtained for classical capacity by Reshetnyak [Res], Martio [Mar], and Maz'ya [Maz].

In joint work with V. Maz'ya (see [CoMa]) we proved the finite superadditivity property of the $p,q$-capacity when $q < \infty$. (See [CoMa, Theorem 3.2, vii) and viii)]). We have the following theorem:

**Theorem 11.** *Suppose $1 < p < \infty$ and $1 \le q < \infty$. Suppose $\Omega_i, \ldots, \Omega_k$ are $k$ pairwise disjoint open sets and $K_i$ are compact subsets of $\Omega_i$ for $i = 1, \ldots, k$.*

(vii) *Suppose $1 \le q \le p$. Then*

$$\mathrm{cap}_{p,q}(\cup_{i=1}^k K_i, \cup_{i=1}^k \Omega_i) \ge \sum_{i=1}^k \mathrm{cap}_{p,q}(K_i,\Omega_i).$$

(viii) *Suppose $p < q < \infty$. Then*

$$\mathrm{cap}_{p,q}(\cup_{i=1}^k K_i, \cup_{i=1}^k \Omega_i)^{q/p} \geq \sum_{i=1}^k \mathrm{cap}_{p,q}(K_i, \Omega_i)^{q/p}.$$

In the same paper we also presented integral conductor inequalities connecting the Lorentz $p, q$-(quasi)norm of a gradient of a function to a one-dimensional integral of the $p, q$-capacitance of the conductor between two level surfaces of the same function. These inequalities generalize an inequality obtained by V. Maz'ya in the case of the Sobolev norm. Such conductor inequalities lead to necessary and sufficient conditions for Sobolev-Lorentz type inequalities involving two arbitrary measures.

Specifically, we proved the inequalities

$$\int_0^\infty \mathrm{cap}(\overline{M_{at}}, M_t) d(t^p) \leq c(a, p, q) ||\nabla f||^p_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)} \text{ when } 1 \leq q \leq p \tag{6}$$

and

$$\int_0^\infty \mathrm{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p} d(t^q) \leq c(a, p, q) ||\nabla f||^q_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)} \text{ when } p < q < \infty \tag{7}$$

for all $f \in Lip_0(\Omega)$. Here $Lip_0(\Omega)$ is the set of all Lipschitz functions compactly supported in the open set $\Omega \subset \mathbf{R}^n$, while $M_t$ is the set $\{x \in \Omega : |f(x)| > t\}$ with $t > 0$.

The proof of (6) and (7) is based on the superadditivity of the $p, q$-capacity.

From (6) and (7) we derived necessary and sufficient conditions for certain two-weight inequalities involving Sobolev-Lorentz norms. Specifically, let $\mu$ and $\nu$ be two locally finite nonnegative measures on $\Omega$ and let $p, q, r, s$ be real numbers such that $1 < s \leq \max(p, q) \leq r < \infty$ and $q \geq 1$. We characterized the inequality

$$||f||_{L^{r,\max(p,q)}(\Omega,\mu)} \leq A \left( ||\nabla f||_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)} + ||f||_{L^{s,\max(p,q)}(\Omega,\nu)} \right) \tag{8}$$

restricted to functions $f \in Lip_0(\Omega)$ by requiring the condition

$$\mu(g)^{1/r} \leq K(\mathrm{cap}_{p,q}(\overline{g}, G)^{1/p} + \nu(G)^{1/s}) \tag{9}$$

to be valid for all open bounded sets $g$ and $G$ subject to $\overline{g} \subset G$, $\overline{G} \subset \Omega$. When $n = 1$ inequality (8) becomes

$$||f||_{L^{r,\max(p,q)}(\Omega,\mu)} \leq A \left( ||f'||_{L^{p,q}(\Omega, m_1)} + ||f||_{L^{s,\max(p,q)}(\Omega,\nu)} \right). \tag{10}$$

The requirement that (10) be valid for all functions $f \in Lip_0(\Omega)$ when $n = 1$ is shown to be equivalent to the condition

$$\mu(\sigma_d(x))^{1/r} \leq K(\tau^{(1-p)/p} + \nu(\sigma_{d+\tau}(x))^{1/s}) \tag{11}$$

whenever $x$, $d$ and $\tau$ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. Here $\sigma_d(x)$ denotes the open interval $(x - d, x + d)$ for every $d > 0$.

In [Cos4] I compared Hausdorff measures and Sobolev relative capacity. If $(X, d, \mu)$ is a proper and unbounded doubling metric measure space that supports a weak $(1, p)$-Poincaré inequality, and $K$ is a compact subset of the bounded open set $\Omega \subset X$, one defines the *relative Sobolev $p$-capacity* of the pair $(K, \Omega)$ by

$$\mathrm{cap}_p(K, \Omega) = \inf \left\{ \int_\Omega g_u^p d\mu : u \in Lip_0(\Omega) \text{ with } u \geq 1 \text{ on a neighborhood of } K \right\},$$

where $g_u$ is a minimal $p$-weak upper gradient of $u$. For the theory of $p$-weak upper gradients and metric space analysis, see for example [Hei], [Sha] and the references therein. We extended this definition to arbitrary sets $E \subset \Omega$. We prove that this capacity is a Choquet set function.

Under the additional assumption that the space $(X, d, \mu)$ has an "upper dimension" $Q$ for some $p \leq Q < \infty$, we obtain lower estimates of the relative Sobolev $p$-capacities in terms of the Hausdorff content associated with continuous gauge functions $h$ that are doubling (that is, there exists a constant $C>1$ such that $h(2t)\le Ch(t)$ for all $t>0$) and satisfy the decay condition

$$\int_0^1 \left(\frac{h(t)}{t^{Q-p}}\right)^{1/p}\frac{dt}{t}<\infty. \tag{12}$$

Specifically, we have (see [Cos4, Theorem 4.4]):

**Theorem 12.** *Suppose $1<p\le Q<\infty$. Let $(X,d,\mu)$ be a proper and unbounded doubling metric measure space that supports a weak $(1,p)$-Poincaré inequality. Suppose that $h:[0,\infty)\to[0,\infty)$ is a doubling homeomorphism satisfying (12). Suppose also that there exists a constant $C_\mu>0$ such that $\mu(B(x,t))\ge C_\mu^{-1}t^Q$ for all $t>0$ and $x\in X$. Then there exists a positive constant $C_1$ depending only on the doubling constant of $h$, on $p$ and on data of $X$ such that*

$$\Lambda_h^\infty(E\cap B(x,r))\le C_1\left(\int_0^{4r}\left(\frac{h(t)}{t^{Q-p}}\right)^{1/p}\frac{dt}{t}\right)^p \operatorname{cap}_p(E\cap B(x,r),B(x,2r))$$

*for every $E\subset X$, every $x\in X$ and every $r>0$.*

This generalized a well-known result obtained by Martio [Mar], Maz'ya [Maz] and Reshetnyak [Res] in $\mathbf{R}^n$.

In the metric setting, in joint work with M. Miranda Jr. we studied the Newtonian Lorentz metric spaces $N^{1,L^{p,q}}(X,\mu)$. These spaces are the generalization of the Newtonian spaces that were studied by Shanmugalingam in [Sha]. We also studied the $p,q$-modulus of families of rectifiable curves and the global $p,q$-capacities associated with these spaces. Under some additional assumptions (namely, the metric space $(X,d)$ carries a doubling measure $\mu$ and satisfies a certain weak $(1,L^{p,q})$ Poincaré inequality) we showed that when $q$ is in $[1,p]$ the Lipschitz functions are dense in these spaces. Specifically, we have (see [CoMi1, Theorem 6.9]):

**Theorem 13.** *Let $1\le q\le p<\infty$. Suppose that $(X,d,\mu)$ is a doubling metric measure space that carries a weak $(1,L^{p,q})$-Poincaré inequality. Then the Lipschitz functions are dense in $N^{1,L^{p,q}}(X,\mu)$.*

Moreover, under the hypotheses of Theorem 13 we showed that the $p,q$-capacity is Choquet provided that $1<q\le p$. We also provided a counterexample for the density result in the Euclidean setting when $p$ is in $(1,n]$ and $q=\infty$.

In [Cos7] we studied the Sobolev-Lorentz spaces and the different inclusions among them in the Euclidean setting for $n\ge 1$.

**Theorem 14.** (See [Cos7, Theorem 3.5].) *Let $n\ge 1$ be an integer. Let $0<\alpha\le 1$ and $r>0$. Suppose $1<p<\infty$ and $1\le q_1<q_2\le\infty$.*

*We define*

$$f_{rad,r,\alpha,p}:[0,r)\to[0,\infty], f_{rad,r,\alpha,p}(t)=\int_t^r u_{rad,r,\alpha,p}(s)ds, \tag{13}$$

*where*

$$u_{rad,r,\alpha,p}(t)=(\Omega_n t^n)^{-\frac{1}{p}}\ln^{-\alpha}\left(\frac{r^n e^{p\alpha}}{t^n}\right). \tag{14}$$

*We also define*

$$v_{r,\alpha,n,p}:B(0,r)\to[0,\infty], v_{r,\alpha,n,p}(x):=f_{rad,r,\alpha,p}(|x|). \tag{15}$$

*Then*

(i) *$v_{r,\alpha,n,p}\in C^\infty(B^*(0,r))$ and*

$$\nabla v_{r,\alpha,n,p}(x)=f'_{rad,r,\alpha,p}(|x|)\frac{x}{|x|} \text{ for all } x\in B^*(0,r).$$

(ii) *$|\nabla v_{r,\alpha,n,p}(x)|=u_{rad,r,\alpha,p}(|x|)$ for all $x\in B^*(0,r)$, where $u_{rad,r,\alpha,p}$ is the function defined in (14).*

(iii) $\lim_{x \to y} v_{r,\alpha,n,p}(x) = 0$ *for all* $y \in \partial B(0,r)$.
(iv) *If* $p > n$, *then* $v_{r,\alpha,n,p}$ *is continuous in* $B(0,r)$.
(v) *If* $1 < p \le n$, *then* $v_{r,\alpha,n,p}$ *is unbounded on* $B(0,r)$.
(vi) $|\nabla v_{r,\alpha,n,p}| \in L^{p,q_2}(B(0,r)) \setminus L^{p,q_1}(B(0,r))$ *if* $1 \le q_1 \le \frac{1}{\alpha} < q_2 \le \infty$.

The following theorem shows, among other things, that $H^{1,(p,\infty)}(\Omega)$ is strictly included in $W^{1,(p,\infty)}(\Omega)$; it also shows that the spaces $H_0^{1,(p,\infty)}(\Omega)$, $H^{1,(p,\infty)}(\Omega)$, and $W^{1,(p,\infty)}(\Omega)$ are not reflexive.

**Theorem 15.** (See [Cos7, Theorem 4.8].) *Let* $\Omega \subset \mathbf{R}^n$ *be an open set, where* $n \ge 1$ *is an integer and let* $y$ *be a point in* $\Omega$. *Suppose* $1 < p < \infty$.
(i) $H^{1,(p,\infty)}(\Omega)$ *is strictly included in* $W^{1,(p,\infty)}(\Omega) \cap H^{1,(p,\infty)}(\Omega \setminus \{y\})$.
(ii) $H^{1,(p,q)}(\Omega \setminus \{y\})$ *is strictly included in* $H^{1,(p,\infty)}(\Omega \setminus \{y\})$ *whenever* $1 \le q < \infty$.
(iii) *The spaces* $H_0^{1,(p,\infty)}(\Omega)$, $H^{1,(p,\infty)}(\Omega)$, *and* $W^{1,(p,\infty)}(\Omega)$ *are not reflexive.*

Next theorem shows that $W^{1,(p,q_1)}(\Omega)$ is stricly included in $W^{1,(p,q_2)}(\Omega)$ whenever $1 < p < \infty$ and $1 \le q_1 < q_2 \le \infty$.

**Theorem 16.** (See [Cos7, Theorem 4.13].) *Let* $n \ge 1$ *be an integer and* $r > 0$ *be a positive number. Suppose* $1 < p < \infty$ *and* $1 \le q_1 < q_2 \le \infty$. *Let* $\alpha$ *be a number in* $(0,1]$ *such that* $1 \le q_1 \le \frac{1}{\alpha} < q_2 \le \infty$. *Let* $v_{r,\alpha,n,p} : B(0,r) \to [0,\infty]$ *be the function defined in* (15).
*Then*
(i) $v_{r,\alpha,n,p} \in H_0^{1,(p,q_2)}(B(0,r)) \setminus H^{1,(p,q_1)}(B(0,r))$.
(ii) $v_{r,\alpha,n,p} \in H^{1,(p,q_2)}(B^*(0,r)) \setminus H^{1,(p,q_1)}(B^*(0,r))$.

We also proved (among other things) that if $n = 1$ and $\Omega \subset \mathbf{R}$ is an open interval, then all the functions in $H_0^{1,(p,q)}(\Omega)$ and in $W^{1,(p,q)}(\Omega)$ have representatives that are Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{1}{p}$.

**Theorem 17.** (See [Cos7, Theorem 5.5].) *Suppose* $n = 1 < p < \infty$ *and* $1 \le q \le \infty$. *Let* $\Omega \subset \mathbf{R}$ *be an open set.*

(i) *Suppose that* $\Omega$ *is an interval. If* $u \in W^{1,(p,q)}(\Omega)$, *then there exists a version* $\overline{u} \in C(\overline{\Omega})$ *that is Hölder continuous in* $\overline{\Omega}$ *with exponent* $1 - \frac{1}{p}$ *and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \le C(p,q)||u'||_{L^{p,q}(\Omega)}.$$

(ii) *If* $u \in W_{loc}^{1,(p,q)}(\Omega)$, *then there exists a version* $\overline{u} \in C(\Omega)$ *that is locally Hölder continuous in* $\Omega$ *with exponent* $1 - \frac{1}{p}$ *and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega'}} \le C(p,q)||u'||_{L^{p,q}(\Omega')},$$

*whenever* $\Omega'$ *is an open subinterval of* $\Omega$ *such that* $\Omega' \subset\subset \Omega$. *Moreover, if* $u' \in L^{(p,q)}(\Omega)$ *and* $\Omega$ *is an interval, then* $\overline{u}$ *is Hölder continuous in* $\overline{\Omega}$ *with exponent* $1 - \frac{1}{p}$ *and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \le C(p,q)||u'||_{L^{p,q}(\Omega)}.$$

(iii) *If* $u \in H_0^{1,(p,q)}(\Omega)$, *then there exists a version* $\overline{u} \in C(\overline{\Omega})$ *that is Hölder continuous in* $\overline{\Omega}$ *with exponent* $1 - \frac{1}{p}$ *and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \le C(p,q)||u'||_{L^{p,q}(\Omega)}.$$

We also proved (among other things) that if $1 < n < p < \infty$ and $1 \le q \le \infty$, then $H_0^{1,(p,q)}(\Omega)$ and the space of functions from $W^{1,(p,q)}(\Omega)$ that are compactly supported in $\Omega$ embed into $C^{0,1-\frac{n}{p}}(\overline{\Omega})$. Since we work with functions in $H_0^{1,(p,q)}(\Omega)$ and with compactly supported functions from $W^{1,(p,q)}(\Omega)$, no regularity assumptions on $\partial\Omega$ are needed. This extends the Morrey embedding theorem to the Sobolev-Lorentz spaces in the Euclidean setting.

**Theorem 18.** (See [Cos7, Theorem 5.6].) *Suppose* $1 < n < p < \infty$ *and* $1 \leq q \leq \infty$. *Let* $\Omega \subset \mathbf{R}^n$ *be open.*

(i) *If* $u \in W^{1,(p,q)}(\Omega)$ *is compactly supported in* $\Omega$, *then* $u$ *has a version* $\overline{u} \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ *and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \leq C(n,p,q)||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)}, \tag{16}$$

*where* $C(n,p,q) > 0$ *is a constant that depends only on* $n, p, q$.

(ii) *If* $u \in W^{1,(p,q)}_{loc}(\Omega)$, *then* $u$ *has a version* $\overline{u}$ *that is locally Hölder continuous in* $\Omega$ *with exponent* $1 - \frac{n}{p}$.

(iii) *If* $u \in W^{1,(p,q)}_{loc}(\mathbf{R}^n)$ *and* $|\nabla u| \in L^{(p,q)}(\mathbf{R}^n)$, *then* $u$ *has a version* $\overline{u} \in C^{0,1-\frac{n}{p}}(\mathbf{R}^n)$ *and*

$$[\overline{u}]_{0,1-\frac{1}{p};\mathbf{R}^n} \leq C(n,p,q)||\nabla u||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)},$$

*where* $C(n,p,q)$ *is the constant from* (16).

(iv) *If* $u \in H^{1,(p,q)}_0(\Omega)$, *then* $u$ *has a version* $\overline{u} \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ *and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \leq C(n,p,q)||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)},$$

*where* $C(n,p,q)$ *is the constant from* (16).

In [Cos8] we studied the Sobolev-Lorentz capacity and its regularity in the Euclidean setting for $n \geq 1$ integer. We extended here our previous results on the Sobolev-Lorentz capacity obtained for $n \geq 2$.

In one of the main results of the paper we obtained sharp estimates for the $n, 1$ relative capacity of the concentric condensers $(\overline{B}(0,r), B(0,1))$ for all $r$ in $[0,1)$, where $n \geq 2$ is an integer. As a consequence we obtained the exact value of the $n, 1$ capacity of a point relative to all its bounded open neighborhoods from $\mathbf{R}^n$ where $n \geq 2$ is an integer. Thus, we improved another previous results of ours. Then we showed that this constant is also the value of the $n, 1$ global capacity of any point from $\mathbf{R}^n$, $n \geq 2$.

**Theorem 19.** (See [Cos8, Theorems 6.2-6.3].) *Let* $n \geq 2$ *be an integer. We denote by* $\Omega_n$ *the Lebesgue measure of the* $n$*-dimensional unit ball.*

(i) *We have*

$$n^n \Omega_n (1-r^n)^{1-n} \leq \mathrm{cap}_{n,1}(\overline{B}(0,r)), B(0,1) \leq n^n \Omega_n \frac{1-r^n}{(1-r)^n}$$

*for every* $0 \leq r < 1$.

(ii) *We have*

$$\mathrm{cap}_{n,1}(\{x\}, \Omega) = n^n \Omega_n$$

*whenever* $x \in \mathbf{R}^n$ *and* $\Omega$ *is a bounded open set in* $\mathbf{R}^n$ *containing* $x$.

(iii) *We have*

$$\mathrm{Cap}_{n,1}(\{x\}) = n^n \Omega_n$$

*whenever* $x \in \mathbf{R}^n$.

Later we gave a new prove of the embedding $H^{1,(n,1)}_0(\Omega) \hookrightarrow C(\overline{\Omega}) \cap L^\infty(\Omega)$, where $\Omega \subset \mathbf{R}^n$ is open and $n \geq 2$ is an integer. This embedding has been proved in the literature, but our method uses the theory of the Sobolev-Lorentz $n, 1$ relative capacity in $\mathbf{R}^n$, $n \geq 2$. Moreover, throughout our prove of this result one can see that the optimal constant of this embedding is related to the aforementioned $n, 1$ capacity of a point relative to any of its bounded neighborhoods from $\mathbf{R}^n$.

**Theorem 20.** (See [Cos8, Theorem 6.4].) *Let* $\Omega \subset \mathbf{R}^n$ *be an open set, where* $n \geq 2$ *is an integer. If* $u \in H^{1,(n,1)}_0(\Omega)$, *then* $u$ *has a version* $u^* \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ *and*

$$||u||_{L^\infty(\Omega)} \leq \frac{1}{n \Omega_n^{1/n}} ||\nabla u||_{L^{n,1}(\Omega;\mathbf{R}^n)}. \tag{17}$$

*Moreover, if* $\Omega \neq \mathbf{R}^n$, *then* $u^* = 0$ *on* $\partial\Omega$.

Then we proved some weak convergence results for bounded sequences of functions in the non-reflexive Sobolev-Lorentz spaces $H^{1,(p,1)}(\Omega)$ and $H_0^{1,(p,1)}(\Omega)$, where $\Omega \subset \mathbf{R}^n$ is open, $n \geq 1$ is an integer and $1 < p < \infty$.

In the last section of the paper we relied on these aforementioned weak convergence results to prove that the relative and the global $(p,1)$ and $p,1$ capacities are Choquet whenever $1 \leq n < p < \infty$ or $1 < n = p < \infty$.

## 2. Future research

We mention some possible directions for future research in the operator aspects of function theory.

If the constant $C_{n,\sigma,p,\delta}(g)$ in Theorem 1 is independent of $p$, that would have a consequence toward the still open corona problem for $H^\infty(\mathbf{B}_n)$, an obvious problem for future study.

**Conjecture 21.** *Given $g_1, ..., g_N \in H^\infty(\mathbf{B}_n)$ satisfying $\sum_{j=1}^N |g_j(z)| \geq 1$, there are $f_1, ..., f_N \in \mathcal{B}(\mathbf{B}_n)$ satisfying $\sum_{j=1}^N g_j(z) f_j(z) = 1$. Since the Bloch space $\mathcal{B}(\mathbf{B}_n)$ is a proper subset of $BMO(\mathbf{B}_n)$, the independence of $C_{n,\sigma,p}$ in terms of $p$ would improve on previous results obtained by Varopoulos* [Var].

We are also investigating an extension of our corona theorem to certain bounded domains in $\mathbf{C}^n$.

In joint work with R. Rochberg, E. Sawyer and B. Wick I am pursuing the characterization of interpolating sequences for Hilbert function spaces, including $H_n^2$. A widely recognized conjecture of K. Seip is that for Hilbert function spaces $\mathcal{H}_k$ with a complete Nevanlinna-Pick kernel $k(z,w)$, a sequence $Z = \{z_j\}_{j=1}^\infty$ is interpolating for the multiplier algebra $M_{\mathcal{H}_k}$ if and only if $Z$ is separated and the measure

$$d\mu_Z = \sum_{j=1}^\infty \left\| k_{z_j} \right\|_{\mathcal{H}_k}^{-2} \delta_{z_j}$$

is a Carleson measure for $\mathcal{H}_k$.

In connection with the area of analysis on metric spaces, in a series of individual articles I will try to obtain exact values for the $p,q$-capacity of the condensers of type $(B(0,r), B(0,R))$ in the Euclidean setting.

I will be studying jointly with M. Miranda Jr. Newtonian Lorentz metric spaces with zero boundary values and the associated $p,q$-capacities. It is interesting to study the subadditivity and superadditivity of these capacities and to find out whether these capacities are Choquet or not. The case $q = \infty$ might pose some extra challenges for the $p,q$-capacity since the weak $L^p$ spaces are nonreflexive and the weak $L^p$ norm is not superadditive. The case $p < q < \infty$ is more challenging than the case $1 \leq q \leq p$ when looking at the density of Lipschitz functions in these Newtonian Lorentz spaces. I would also like to study such features as fine behavior of functions, Lebesgue points, and quasicontinuity. I would also like to obtain sharp estimates for the $p,q$-capacity of the condensers of type $(B(x,r), B(x,R))$ in the metric setting.

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